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Note on Diffusion of Vapor into Flowing Gas

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The theory of diffusion of vapor between the walls of a tube and a stream of gas is applied to a generator of known humidities. The rate of approach to equilibrium is evaluated for gas velocities in the laminar flow range. The effect of pressure drop is examined.

Key words: Diffusion; equilibrium; evaporation; humidity; humidity generator; laminar flow; saturation; water vapor.

The apparatus described by Greenspan, [1] [1] as well as others used to generate known humidities, includes a long tube of negligible curvature, down which a carrier gas flows. The walls of the tube are coated with the solid or liquid phase, and the vapor diffuses between the walls and the carrier gas. The length of the tube should be such that the gas should be fully saturated with the vapor at the outlet, or at least, the departure from full saturation should be well known. Precautions should be taken against entrainment of water droplets or ice particles.

With the addition of terms which represent the transport of the vapor, the differential equation for diffusion becomes:

$$D\,\nabla^2 c - \partial c/\partial t - \mathbf{v}\cdot\nabla c - c\ \mathrm{Div}\ \mathbf{v} = 0$$

where c is the vapor concentration, $\mathbf{v}$ the velocity of the carrier gas, and D the coefficient of diffusion for the vapor in the carrier gas. We will consider the case of laminar flow, far enough down the tube that the entry transients are negligible. We will take the channel as circular, of radius a. In the steady state the flow of the gas will be parallel to the axis and the axial velocity at distance r from the axis given by $v=v_0(1-r^2/a^2)$ where v_0 is the velocity at the axis. Div $\mathbf{v}$ represents the expansion of the gas in consequence of the pressure drop in the tube. If this is small, the acceleration of the gas will produce negligible distortion of the parabolic flow pattern, so that Div $\mathbf{v}=-(v_0/p)(1-r^2/a^2)\ dp/dz$. After the start-up transients have subsided, the time derivatives are zero. We can now write the differential equation as:

$$\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}+\frac{\partial^2 c}{\partial z^2}+\frac{v_0}{D}\left(1-\frac{r^2}{a^2}\right)\left[\frac{c}{p}\frac{dp}{dz}-\frac{\partial c}{\partial z}\right]=0$$

where z is the axial coordinate.

The importance of the velocity effects can be measured by a dimensionless number $b=av_0/D$. For water vapor in air, b is about 1.5 times the Reynolds number.

Let us seek a general solution of the differential equation as the sum of a set of terms each of which can be separated into factors dependent on the axial and on the radial coordinates. Near the outlet the solution will be shown to be dominated by a term which includes the expansion of the carrier gas due to the pressure drop. Upstream, terms will be found which contain factors which decay exponentially with respect to the axial coordinate. We will be concerned particularly with the more persistent of these terms. All terms of the solutions will be finite within the tube, i.e., for $r<a$ and symmetrical about the axis at $r=0$. At the wall ($r=a$) the solution has a value independent of z; for the terms which decay exponentially the value is zero. Let us try the sum of a number of terms of the form $\phi e^{-\alpha z/a}$, where ϕ is a function only of r, say $\phi=\sum_0^\infty A_n(r^2/a^2)^n$. The requirements of symmetry and finite values are met by restricting n to positive integers. Putting this in the differential equation we get

$$\left[a^2\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)+\alpha^2\phi + b\left(1-\frac{r^2}{a^2}\right)(\beta+\alpha)\phi\right]e^{-\alpha z/a}=0$$

where $\beta=(a/b)(dp/dz)$ is the (taken as constant) fractional loss of pressure in a distance equal to the radius. Substituting for ϕ and performing the differentiations the following relations between coefficients are obtained:

$$4A_1=-[b(\beta+\alpha)+\alpha^2]A_0$$

[1] Figures in brackets indicate the literature references at the end of this paper.

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$$4n^2 A_n = -[b(\beta+\alpha)+\alpha^2]A_{n-1} + b(\beta+\alpha)A_{n-2}$$

ϕ reduces to zero at $r=a$ for an infinite number of values of α. For $\beta < \alpha \ll b$, the first three of the roots are given by

$$(\alpha+\beta)b = 7.3136,\ 44.608,\ 113.922.$$

The closely related problem of the transfer of heat from a tube to a fluid was examined by Graetz [2] and independently by Nusselt [3]. Their solution was essentially the same as the above, and is summarized in a review article by Drew [4] who also recalculated the roots.

Their interest, and the experimental comparisons, were with relatively short tubes, for which the terms containing the second and third roots were significant, and they omitted the quantities α^2 and β in the differential equation.

Their values for the roots are compared with the values indicated below in the table

Roots of $\phi_j(1)=0$

j	Graetz	Nusselt	Drew	$\sqrt{(\alpha+\beta)b}$
1	2.7043	2.705	2.70436	2.704365
2	6.50	6.66	6.6791	6.67903
3		10.3		10.67337

When ϕ is plotted as a function of the radius the curves corresponding to these roots resemble the Bessel function $J_0(\gamma r/a)$, where $\gamma^2=(\alpha+\beta)b$, for small radii. As $r\rightarrow a$, the curve is flattened somewhat. For $r>a$ the function increases in absolute value without limit. This set of functions can be used to fit boundary conditions, such as the humidity distribution of the gas entering the tube, in a manner similar to the Fourier of Bessel series.

The coefficient α^2 in the differential equation represents diffusion in the axial direction. It reduces the value of the root slightly. The conditions of operation to be evaluated in Greenspan's experiment are a mean velocity of 71 cm/s, a radius of 0.4 cm, and a diffusion coefficient of 0.1 cm/s corresponding to -80 °C at atmospheric pressure. Thus $b\sim 530$, $\alpha\sim 0.014$ and the effect is to reduce the value of the first root from $(\alpha+\beta)b=7.3136$ to 7.3125.

The distance required for a term to decrease by a factor of e equals a/α. For Greenspan's experiment this is 29 cm for the first root, 4.7 cm for the second, and 1.85 cm for the third. The decay distances for higher roots are much smaller. If the tube is long enough these terms will decay to negligible values at the outlet, and the associated transfer of material at the wall is substantially complete.

The effect of pressure drop is indicated by the coefficient β in the differential equation. This may be small compared with the smallest value of α considered above, but significant when $\alpha=0$, corresponding to a distribution of concentration which is uniform over the length of the tube. If βb is small so that β^2b^2 can be neglected, only the first three terms of the series are significant and we have

$$A_1 = -\beta b A_0,\ A_2 = (\beta b+\beta^2b^2)A_0/4\ \ldots$$

At radius r

$$C(r) = A_0\left[1-\beta b\,(r^2/a^2)+(\beta b/4)(r^4/a^4)+\beta^2\,(\ldots)\right]$$

At the wall, $r=a$

$$C(a) = A_0\left[1-\beta b+\beta b/4+\beta^2\,(\ldots)\right]$$

Because of the expansion of the carrier gas there will also be a transfer of gas from the wall to the stream at a uniform rate

$$2\pi a D\left(\frac{-\partial C(a)}{\partial r}\right) = 2\pi D A_0 \beta b \text{ per unit length.}$$

At the outlet, if all other terms have decayed to negligible magnitude, the concentration is

$$\frac{C(r)}{C_w} = \frac{A_0[1-\beta b(r^2/a^2)+(\beta b/4)\,(r^4/a^4)-\beta^2(\ldots)]}{A_0[1-\beta b+(\beta b/4)+\beta^2\ldots]}$$

by long division, we get

$$C(r)/C_w = 1+\beta b(1-r^2/a^2)-(\beta b/4)\,(1-r^4/a^4) + \beta^2(\ldots).$$

The quantity of vapor discharged at the outlet is

$$\pi a^2 v_0 \bar{C}/2 = \pi\int_0^{r=a} vC(r)d(r^2)$$

$$= \pi a^2 \int_0^{r=a} v_0(1-r^2/a^2)C(r)d(r^2/a^2)$$

$$= \pi a^2 v_0 C_w \int_0^{r=a} (1-r^2/a^2)\,[1+\beta b(1-r^2/a^2) - (\beta b/4)\,(1-r^4/a^4)+\ldots]d(r^2/a^2)$$

$$= \pi a^2 v_0 C_w\left[\frac{1}{2}+\beta b\int_0^{r=a}\left(1-\frac{2r^2}{a^2}+\frac{r^4}{a^4}\right)d\left(\frac{r^2}{a^2}\right) - \frac{\beta b}{4}\int_0^{r=a}\left(1-\frac{r^2}{a^2}-\frac{r^4}{a^4}+\frac{r^6}{a^6}\right)d\left(\frac{r^2}{a^2}\right)+\ldots\right]$$

$$= \pi a^2 v_0 C_w\left[\frac{1}{2}+\beta b(1-1+1/3) - (\beta b/4)\,(1-1/2-1/3+1/4)\ldots\right]$$

$$= \pi a^2 v_0 C_w(1/2+11\beta b/48+\ldots).$$

Therefore

$$(\bar{C}-C_w)/C_w=11\beta b/24.$$

In the apparatus used by Greenspan the pressure drop was measured to be $\beta=(adp/dz)/p_0=-5.2\times10^{-8}$ at atmospheric pressure so that $\beta b=-2.8\times10^{-5}$. The corresponding error in concentration is $(C-C_w)/C_w=-(11/24)\times2.8\times10^{-5}\sim-1.2\times10^{-5}$.

In the above we have taken the vapor concentration at the interface with the condensed phase to be that in equilibrium at the bath temperature. A more complete discussion would take into account the temperature gradient in the wall and the balance at the interface between the latent heat of evaporation and the thermal conduction in the wall and in the gas stream. It should also consider the effect of the partial pressure of the vapor on the conservation equation. These factors can be neglected in Greenspan's experiment since the partial pressure of vapor is 0.1 percent or less, and a rough calculation indicates that the thermal gradient in the wall will change the decay lengths by less than 1 percent at 0 °C and much less at lower temperatures. They cannot be neglected at higher temperatures. There is also the possibility that the rate of evaporation or condensation is affected by a potential barrier in a surface film.

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